Algebra III Mid Term Exam

September 9 2015

This exam is of 40 marks. There are 10 questions and subquestions, each of 4 marks plus an extra credit question of 4 marks as well. Please read all the questions carefully and do not cheat. Good luck! (40)

1. Let R be a **finite** commutative ring. Show that every **prime** ideal is **maximal**. (4)

2. Let R be a ring in which $a^4 = a$ for all $a \in R$. Is R commutative? Prove your answer. (4)

3a. Prove that the polynomial $X^2 + 2X + 2$ is irreducible in $\mathbb{Z}[X]$. (4)

3b. Is it irreducible in $\mathbb{Q}[X]$? Give reasons for your answer. (4)

4. A module M is said to be **simple** if the only submodules of M are M and $\{0\}$.
a. Give an example of a ring R and a simple R-module over it. (4)

b. Let M be any R-module. Let $\mathrm{End}(M)$ denote the set of R-module homomorphisms $\phi : M \to M$ such that $\phi(ra + sb) = r\phi(a) + s\phi(b)$ where $r, s \in R$ and $a, b \in M$. Show that

1. $\mathrm{End}(M)$ is a ring under addition and composition. (4)
2. If M is a simple, non-trivial R module then $\mathrm{End}(M)$ is a **division ring** (that is, all non-zero elements are invertible). (4)

5. If $f \in \mathbb{Z}/p\mathbb{Z}[X]$, p a prime, is **irreducible** of degree n. Show that

1. Show that $\mathbb{Z}/p\mathbb{Z}[X]/(f)$ is a field. (4)
2. Does it have finitely many elements? If so how many? (4)

6. Show that there are infinitely many primes of the form $4n + 3$. (4)

7 **(Extra Credit)**. Show that there are infinitely many primes of the form $4n + 1$. (Hint: If $p|(N^2 + 1) \Rightarrow p \equiv 1 \bmod 4$ (Why?)) . (4)